

THE KREĬN-MIL'MAN THEOREM FOR METRIC SPACES WITH A CONVEX BICOMBING

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ABSTRACT. We use bicomblings on arcwise connected metric spaces to give definitions of convex sets and extremal points. These notions coincide with the customary ones in the classes of normed vector spaces and geodesic metric spaces which are convex in the usual sense. A rather straightforward modification of the standard proof of the Kreĭn-Mil'man Theorem yields the result that in a large class of metric spaces every compact convex set is the closed convex hull of its extremal points. The result appears to be new even for CAT(0)-spaces.

Definition. Let (X, d) be a metric space. A *convex bicombling* on X is a map $X \times X \rightarrow C([0, 1], X)$, $(x, y) \mapsto [x, y](\cdot)$ satisfying

- (i) For all $x, y \in X$, one has $[x, y](0) = x$ and $[x, y](1) = y$. Moreover, $[x, x] \equiv x$.
- (ii) For all $x, y, x', y' \in X$ the function $t \mapsto d([x, y](t), [x', y'](t))$ is a convex function on $[0, 1]$.

Examples. We are interested in the following two special cases:

- (i) Let (X, d) be a *convex* metric space in the sense that any two points of X are connected by a geodesic and that the inequality

$$d(c(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1))$$

holds for all linearly parameterized geodesics $c, c' : [0, 1] \rightarrow X$ and all $t \in [0, 1]$. It is easy to see that such a space is uniquely geodesic and that the map associating to $x, y \in X$ the unique linearly parameterized geodesic $[x, y]$ connecting x to y is a convex bicombling. Observe that the class of convex metric spaces contains in particular all CAT(0)-spaces.

- (ii) If $(X, \|\cdot\|)$ is a normed vector space then by setting $[x, y](t) = (1-t)x + ty$ for all $x, y \in X$ one obtains a convex bicombling on X .

Throughout this note, X stands as a shorthand for a metric space $(X, d, [\cdot, \cdot])$ with a convex bicombling. The choice of a bicombling in a metric space yields notions of convexity and extremity as follows: A subset $C \subset X$ is called *convex* if $x, y \in C$ implies $[x, y] \subset C$. A function $\phi : A \rightarrow \mathbb{R}$ on a subset A of X is called *convex* if $t \mapsto \phi([x, y](t))$ is a convex function on the interval $[0, 1]$ for all $x, y \in A$ with $[x, y] \subset A$. The *(closed) convex hull* of a subset A of X is the smallest (closed) convex set containing A .

Definition. Let $C \subset X$ be a convex set. A *closed* non-empty subset $E \subset C$ is called *extremal* if for all $x, y \in C$ for which there exists some $t \in (0, 1)$ with $[x, y](t) \in E$ one has $[x, y] \subset E$. A point $p \in C$ is called *extremal* if $\{p\}$ is an extremal set.

We can now state the Kreĭn-Mil'man Theorem:

Theorem. *If C is a compact and convex subset of a metric space X with convex bicombling then C is the closed convex hull of the set of its extremal points.*

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Consider the family of extremal subsets of C and order it by inclusion. This family is non-empty since it contains C and by compactness of C the intersection of a decreasing family of extremal sets is non-empty, hence it is an extremal set. Zorn's lemma applies and yields:

Lemma 1. *Every extremal subset of C contains a minimal extremal subset.* \square

The next step is to prove:

Lemma 2. *Every minimal extremal subset is a one-point set.*

Before giving the proof, we record the following:

Lemma 3. *Let E be an extremal subset of C and let $\phi : E \rightarrow \mathbb{R}$ be continuous and convex. Then $E_\phi = \{e \in E : \phi(e) = \max \phi\}$ is an extremal subset of C .*

Proof of Lemma 3. Since E is compact and ϕ is continuous, the set E_ϕ is closed and non-empty. Let $x, y \in C$ and $t \in (0, 1)$ be such that $[x, y](t) \in E_\phi$. Since E is extremal, we have $[x, y] \subset E$. Now notice that $t \mapsto \phi([x, y](t))$ is convex on $[0, 1]$ (by convexity of the bicombing) and assumes its maximum at some point $t \in (0, 1)$, hence it is constant and thus $[x, y] \subset E_\phi$. \square

Proof of Lemma 2. Let E be a minimal extremal subset of C and let $e \in E$. Consider the function $\phi : x \mapsto d(x, e)$ which is convex by the convexity of the bicombing and the assumption $[x, x] \equiv x$. By Lemma 3 the set E_ϕ is an extremal subset of C . If there exists a point $e' \in E$ distinct from e , then ϕ is not constant, hence E_ϕ is a proper extremal subset of E contradicting the minimality of E . \square

We need one more fact before we can finish the proof of the Kreĭn-Mil'man Theorem.

Lemma 4. *For a compact and convex set $K \subset X$ put $d_K(x) = \min_{k \in K} d(k, x)$. The function $d_K : X \rightarrow \mathbb{R}$ is continuous and convex.*

Proof. It follows from the triangle inequality that d_K is 1-Lipschitz, hence continuous. Let $x, y \in X$ be arbitrary points and pick points $\bar{x}, \bar{y} \in K$ such that $d(x, \bar{x}) = d_K(x)$ and $d(y, \bar{y}) = d_K(y)$. By convexity of the bicombing the function $t \mapsto d([x, y](t), [\bar{x}, \bar{y}](t))$ is convex and by convexity of K , we have $[\bar{x}, \bar{y}] \subset K$, so $d_K([x, y](t)) \leq d([x, y](t), [\bar{x}, \bar{y}](t)) \leq (1-t)d(x, \bar{x}) + td(y, \bar{y}) = (1-t)d_K(x) + td_K(y)$ and it follows that d_K is convex. \square

Proof of the Theorem. Let K be the closed convex hull of the set of extremal points of C . By Lemma 1 and Lemma 2, K is non-empty and by convexity of C we have $K \subset C$. By Lemma 4 the function $\phi = d_K$ is continuous and convex on C , so by Lemma 3 E_ϕ is an extremal set, and it contains an extremal point of C by Lemma 1 and Lemma 2, so $E_\phi \cap K$ is non-empty. If there existed a point $p \in C \setminus K$ then ϕ would be non-constant, so E_ϕ would have to be disjoint from K , a contradiction. \square

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